Assignment 10

This homework is due *Thursday* Nov 13.

There are total 26 points in this assignment. 23 points is considered 100%. If you go over 23 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 5.3, 5.4 in Bartle–Sherbert.

(1) [2pt] (4.2.13a, take two) Fix Problem 5a of Assignment 8 so that the values $\lim_{x\to 1} g(f(x))$ and $g(\lim_{x\to 1} f(x))$ are different.
[Reminder: Problem 5b of Assignment 8 asked the following.
Functions f and g are defined on $\mathbb{R}$ by $f(x) = x + 1$ and $g(x) = 2$ if $x \neq 1$ and $g(1) = 0$.
(a) Find $\lim_{x\to 1} g(f(x))$ and compare with the value of $g(\lim_{x\to 1} f(x))$.)]

(2) [2pt] (5.3.1) Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for all $x \in I$. Prove that there is a number $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in I$.

(3) [2pt] (Part of 5.3.5) Show that the polynomial $x^4 + 7x^3 - 9$ has at least two real roots.

(4) [3pt] (5.3.6) Let f be continuous on the interval $[0, 1]$ to $\mathbb{R}$ and such that $f(0) = f(1)$. Prove that there exists a point $c \in [0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$. (*Hint:* Consider $g(x) = f(x) - f(x + \frac{1}{2})$.)
Note. Therefore, there are, at any time, antipodal points on the earth's equator that have the same temperature.

(5) [4pt] (5.3.13) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $\lim_{x\to-\infty} f = \lim_{x\to+\infty} f = 0$. Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or a minimum on $\mathbb{R}$. Give an example to show that both a maximum and a minimum need not be attained. (*Hint:* Pick M large enough and inspect how f behaves on an interval $[-M, M]$, on $\mathbb{R}\setminus[-M, M]$.)

(6) (a) [3pt] (5.3.11) Let $I = [a, b]$, let $f : I \to \mathbb{R}$ be continuous on I, and assume that $f(a) < 0$, $f(b) > 0$. Let $W = \{x \in I : f(x) < 0\}$, and let $w = \sup W$. Prove that $f(w) = 0$. (This provides an alternate proof of Location of Roots Theorem.)
(b) [1pt] Why the same reasoning does not necessarily work if both $f(a) > 0, f(b) > 0$? (That is, find a precise place in the construction above that doesn't go through in such case.)

(7) [2pt] (5.4.2) Show that function $f(x) = 1/x^2$ is uniformly continuous on $A = [1, \infty)$, but that it is not uniformly continuous on $B = (0, \infty)$.

(8) (5.4.3) Use the Nonuniform Continuity Criterion to show that the following functions are not uniformly continuous on the given sets.
(a) [2pt] $f(x) = x^2$, $A = [0, \infty)$.
(b) [2pt] $g(x) = \sin(1/x)$, $B = (0, \infty)$.

(9) [3pt] (5.4.6) Show that if f and g are uniformly continuous on $A \subseteq \mathbb{R}$, and if they are *both* bounded on A, then their product fg is uniformly continuous on A.